Selected Topics of Mathematical Statistics
Convergence of Moments & Uniform Integrability

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Definition 1

A sequence of rv $\{Y_n\}$ is **uniformly integrable** (ui) if

$$lim_{c\to\infty} \sup_n \mathrm{E}\{\,|Y_n|\,\mathrm{I}(|Y_n| > c)\} = 0 \tag{1}$$

Definition 2

A sequence of set functions $\{Q_n\}$ defined on $\mathscr{A}$ is **uniformly absolutely continuous** (uac) wrt a measure P on $\mathscr{A}$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathrm{P}(A) < \delta > \sup_n |Q_n(A)| < \varepsilon \tag{2}$$

Definition 3

The sequence $\{Q_n\}$ is **equicontinuous** (ec) at ϕ if, given $\varepsilon > 0$ and a sequence $\{A_n\}$ in $\mathscr{A}$ decreasing to ϕ, there exists M such that

$$m > M > \sup_n |Q_n(A_m)| < \varepsilon \tag{3}$$

Later on $Q_{n,r}(A) = \int_A |X_n|^r \, d\mathsf{P}$, wrt a measure P

Lemma 1

i) *ui of* $\{Y_n\}$ *on* $(\Omega, \mathscr{A}, \mathsf{P})$ *is equivalent to*

a) $sup_n \mathsf{E}|Y_n| < \infty$ *and* b) $\{Q_{n,1}\}$ *are uac wrt* P

ii) *ui of* $\{Y_n\}$ *is that for some* $\varepsilon > 0$

$$sup_n \mathsf{E}|Y_n|^{1+\varepsilon} < \infty > \text{ is ui}$$

iii) *Sufficient for ui of* $\{Y_n\}$ *is that there is a rv* Y *such that*

$\mathsf{E}|Y| < \infty$ *and*

$$\mathsf{P}(|Y_n| \geq y) \leq \mathsf{P}(|Y| \geq y), \text{ all } n \geq 1, \text{ all } y > 0$$

iv) *For set functions* $Q_{n,1}$ *each absolutely continuous wrt* P, *ec at*

ϕ *implies uac wrt* P

Theorem 1

Let $X_n \xrightarrow{\mathscr{L}} X$ and $\{X_n^r\}$ is ui, where $r > 0$. Then $\mathsf{E}|X|^r < \infty$, $\lim_n \mathsf{E}(X_n^r) = \mathsf{E}(X^r)$, and $\lim_n \mathsf{E}|X_n|^r = \mathsf{E}|X|^r$.

Proof of Theorem 1 Let $X \sim F$ and $\varepsilon > 0$. Choose c such that $\pm c$ are continuity points of F, and by ui,

$$sup_n \mathsf{E}\{|X_n|^r \,\mathtt{I}(|X_n| \geq c)\} < \varepsilon.$$

For any $d > c$ such that $\pm d$ are also continuity points of F, we obtain from the second theorem of Helly that

$$\lim_{n\to\infty} \mathsf{E}\{|X_n|^r \mathtt{I}(c \leq |X_n| \leq d)\} = \mathsf{E}\{|X|^r \mathtt{I}(c \leq |X| \leq d)\}.$$

It follows that $\mathsf{E}\{|X|^r \mathrm{I}(c \leq |X| \leq d)\} < \varepsilon$ for all such choices of d.

For $d \to \infty$, $\mathsf{E}\{|X|^r \mathrm{I}(|X| \geq c)\} < \varepsilon$, whence $\mathsf{E}|X|^r < \infty$.

Now, for the same c as above, write

$$
\begin{aligned}
|\mathsf{E}\left(X_n^r\right) - \mathsf{E}\left(X^r\right)| \;\leq\; & |\mathsf{E}\{X_n^r \mathrm{I}(|X_n| \leq c)\} - \mathsf{E}\{X^r \mathrm{I}(|X| \leq c)\}| \\
& + \underbrace{\mathsf{E}\{|X_n|^r \mathrm{I}(|X_n| > c)\}}_{<\varepsilon} + \underbrace{\mathsf{E}\{|X|^r \mathrm{I}(|X| > c)\}}_{<\varepsilon}.
\end{aligned}
$$

By the Helly theorem again, the first term on the right-hand side tends to 0 as $n \to \infty$. Thus $\lim_n \mathsf{E}\left(X_n^r\right) = \mathsf{E}\left(X^r\right)$. Similarly $\lim_n \mathsf{E}|X_n|^r = \mathsf{E}|X|^r$.

Theorem 2

$Let\ X_n \xrightarrow{rth} X\ and\ \mathsf{E}|X|^r < \infty$

$\succ \lim_n \mathsf{E}(X_n^r) = \mathsf{E}(X^r)\ and\ \lim_n \mathsf{E}|X_n|^r = \mathsf{E}|X|^r.$

Theorem 3

$Let\ X_n \xrightarrow{\mathrm{P}} X\ and\ either$

i $\mathsf{E}|X|^r < \infty\ and\ \{X_n^r\}\ ui,\ or$

ii $\sup_n \mathsf{E}|X|^r < \infty\ and\ \{Q_n\}\ are\ ec\ at\ \phi$

$\succ \lim_n \mathsf{E}(X_n^r) = \mathsf{E}(X^r)\ and\ \lim_n \mathsf{E}|X_n|^r = \mathsf{E}|X|^r.$

Theorem 4

$Let\ X_n \xrightarrow{a.s.} X.\ If\ \overline{\lim}_n \mathsf{E}|X_n|^r \le \mathsf{E}|X|^r < \infty$

$\succ \lim_n \mathsf{E}(X_n^r) = \mathsf{E}(X^r)$ and $\lim_n \mathsf{E}|X_n|^r = \mathsf{E}|X|^r.$

Lemma 2

$$Let\ X_n \xrightarrow{\mathscr{L}} X\ and\ \lim_n \mathsf{E}\,|X_n|^r = \mathsf{E}\,|X|^r < \infty \succ \{X_n\}\ is\ ui$$

Proof of Theorem 2 For $0 < r \leq 1$, apply $|x+y|^r \leq |x|^r + |y|^r$ to write $||x|^r - |y|^r| \leq |x-y|^r$ and thus

$$|\mathsf{E}\,|X_n|^r - \mathsf{E}\,|X|^r| \leq \mathsf{E}\,|X_n - X|^r.$$

For $r > 1$, apply Minkowski's inequality to obtain

$$|(\mathsf{E}\,|X_n|^r)^{1/r} - (\mathsf{E}\,|X|^r)^{1/r}| \leq (\mathsf{E}\,|X_n - X|^r)^{1/r}.$$

In either case, $\lim_n \mathsf{E}\,|X_n|^r = \mathsf{E}\,|X|^r < \infty$ follows Lemma 2, $\{X_n^r\}$ is ui $\succ \lim_n \mathsf{E}(X_n^r) = \mathsf{E}(X^r)$ follows from Theorem 2.

Hermann Minkowski on BBI:

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