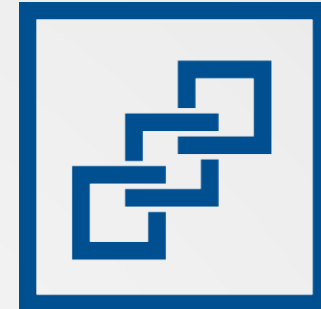
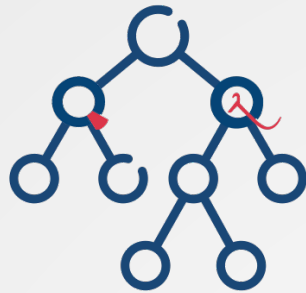




Selected Topics of Mathematical Statistics

Convergence of Moments & Uniform Integrability

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Definition 1

A sequence of rv $\{Y_n\}$ is **uniformly integrable** (ui) if

$$\lim_{c \rightarrow \infty} \sup_n E\{|Y_n| \mathbf{I}(|Y_n| > c)\} = 0 \quad (1)$$

Definition 2

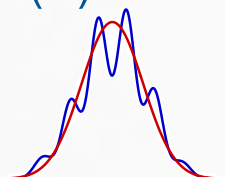
A sequence of set functions $\{Q_n\}$ defined on $\mathcal{A}$ is **uniformly absolutely continuous** (uac) wrt a measure P on $\mathcal{A}$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$P(A) < \delta \Rightarrow \sup_n |Q_n(A)| < \varepsilon \quad (2)$$

Definition 3

The sequence $\{Q_n\}$ is **equicontinuous** (ec) at ϕ if, given $\varepsilon > 0$ and a sequence $\{A_n\}$ in $\mathcal{A}$ decreasing to ϕ , there exists M such that

$$m > M \Rightarrow \sup_n |Q_n(A_m)| < \varepsilon \quad (3)$$



Later on $Q_{n,r}(A) = \int_A |X_n|^r dP$, wrt a measure P

Lemma 1

i) *ui of $\{Y_n\}$ on $(\Omega, \mathcal{A}, P)$ is equivalent to*

a) $\sup_n E|Y_n| < \infty$ and b) $\{Q_{n,1}\}$ are uac wrt P

ii) *ui of $\{Y_n\}$ is that for some $\varepsilon > 0$*

$\sup_n E|Y_n|^{1+\varepsilon} < \infty$ is ui

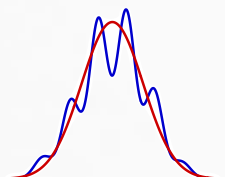
iii) *Sufficient for ui of $\{Y_n\}$ is that there is a rv Y such that*

$E|Y| < \infty$ and

$P(|Y_n| \geq y) \leq P(|Y| \geq y)$, all $n \geq 1$, all $y > 0$

iv) *For set functions $Q_{n,1}$ each absolutely continuous wrt P , ec at*

ϕ implies uac wrt P



Theorem 1

Let $X_n \xrightarrow{\mathcal{L}} X$ and $\{X_n^r\}$ is ui, where $r > 0$. Then $E|X|^r < \infty$,

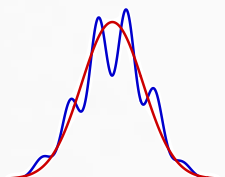
$\lim_n E(X_n^r) = E(X^r)$, and $\lim_n E|X_n|^r = E|X|^r$.

Proof of Theorem 1 Let $X \sim F$ and $\varepsilon > 0$. Choose c such that $\pm c$ are continuity points of F , and by ui,

$$\sup_n E\{|X_n|^r \mathbf{I}(|X_n| \geq c)\} < \varepsilon.$$

For any $d > c$ such that $\pm d$ are also continuity points of F , we obtain from the second theorem of Helly that

$$\lim_{n \rightarrow \infty} E\{|X_n|^r \mathbf{I}(c \leq |X_n| \leq d)\} = E\{|X|^r \mathbf{I}(c \leq |X| \leq d)\}.$$



It follows that $\mathbf{E}\{|X|^r \mathbf{I}(c \leq |X| \leq d)\} < \varepsilon$ for all such choices of d .

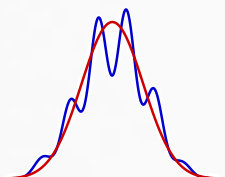
For $d \rightarrow \infty$, $\mathbf{E}\{|X|^r \mathbf{I}(|X| \geq c)\} < \varepsilon$, whence $\mathbf{E}|X|^r < \infty$.

Now, for the same c as above, write

$$\begin{aligned} |\mathbf{E}(X_n^r) - \mathbf{E}(X^r)| &\leq |\mathbf{E}\{X_n^r \mathbf{I}(|X_n| \leq c)\} - \mathbf{E}\{X^r \mathbf{I}(|X| \leq c)\}| \\ &\quad + \underbrace{\mathbf{E}\{|X_n|^r \mathbf{I}(|X_n| > c)\}}_{< \varepsilon} + \underbrace{\mathbf{E}\{|X|^r \mathbf{I}(|X| > c)\}}_{< \varepsilon}. \end{aligned}$$

By the Helly theorem again, the first term on the right-hand side tends to 0 as $n \rightarrow \infty$. Thus $\lim_n \mathbf{E}(X_n^r) = \mathbf{E}(X^r)$. Similarly

$$\lim_n \mathbf{E}|X_n|^r = \mathbf{E}|X|^r.$$



Theorem 2

Let $X_n \xrightarrow{rth} X$ and $E|X|^r < \infty$

> $\lim_n E(X_n^r) = E(X^r)$ and $\lim_n E|X_n|^r = E|X|^r$.

Theorem 3

Let $X_n \xrightarrow{P} X$ and either

i $E|X|^r < \infty$ and $\{X_n^r\}$ ui, or

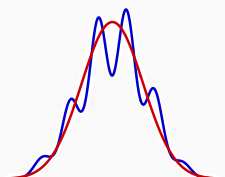
ii $\sup_n E|X|^r < \infty$ and $\{Q_n\}$ are ec at ϕ

> $\lim_n E(X_n^r) = E(X^r)$ and $\lim_n E|X_n|^r = E|X|^r$.

Theorem 4

Let $X_n \xrightarrow{a.s.} X$. If $\overline{\lim}_n E|X_n|^r \leq E|X|^r < \infty$

> $\lim_n E(X_n^r) = E(X^r)$ and $\lim_n E|X_n|^r = E|X|^r$.



Lemma 2

Let $X_n \xrightarrow{\mathcal{L}} X$ and $\lim_n \mathbb{E} |X_n|^r = \mathbb{E} |X|^r < \infty \Rightarrow \{X_n\}$ is ui

Proof of Theorem 2 For $0 < r \leq 1$, apply $|x + y|^r \leq |x|^r + |y|^r$ to write $||x|^r - |y|^r| \leq |x - y|^r$ and thus

$$|\mathbb{E} |X_n|^r - \mathbb{E} |X|^r| \leq \mathbb{E} |X_n - X|^r.$$

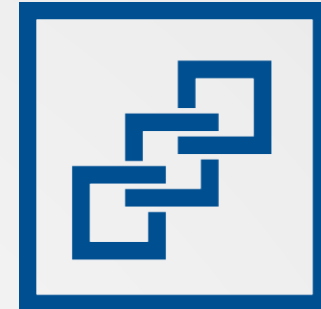
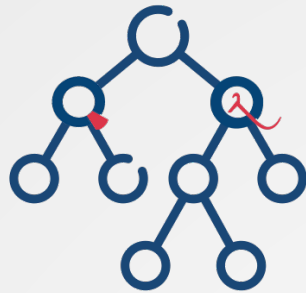
For $r > 1$, apply Minkowski's inequality to obtain

$$|(\mathbb{E} |X_n|^r)^{1/r} - (\mathbb{E} |X|^r)^{1/r}| \leq (\mathbb{E} |X_n - X|^r)^{1/r}.$$

In either case, $\lim_n \mathbb{E} |X_n|^r = \mathbb{E} |X|^r < \infty$ follows Lemma 2, $\{X_n^r\}$ is ui $\Rightarrow \lim_n \mathbb{E}(X_n^r) = \mathbb{E}(X^r)$ follows from Theorem 2.

Hermann Minkowski on BBI:





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